

# Improved dispersive analysis of the scalar form factor of the nucleon

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We present a coupled system of integral equations for the  $\pi\pi \rightarrow \bar{N}N$  and  $\bar{K}K \rightarrow \bar{N}N$   $S$ -waves derived from Roy–Steiner equations for pion–nucleon scattering. We discuss the solution of the corresponding two-channel Muskhelishvili–Omnès problem and apply the results to a dispersive analysis of the scalar form factor of the nucleon fully including  $\bar{K}K$  intermediate states. In particular, we determine the corrections  $\Delta_\sigma$  and  $\Delta_D$ , which are needed for the extraction of the pion–nucleon  $\sigma$  term from  $\pi N$  scattering, and show that the difference  $\Delta_D - \Delta_\sigma = (-1.8 \pm 0.2) \text{ MeV}$  is insensitive to the input  $\pi N$  parameters.

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## 1. Introduction

The pion–nucleon  $\sigma$  term  $\sigma_{\pi N}$  measures the contribution of the light quarks to the nucleon mass  $m$ , and is directly related to the form factor of the scalar current

$$\sigma(t) = \frac{1}{2m} \langle N(p') | \hat{m}(\bar{u}u + \bar{d}d) | N(p) \rangle, \quad t = (p' - p)^2, \quad \hat{m} = \frac{m_u + m_d}{2}, \quad (1.1)$$

at vanishing momentum transfer  $\sigma(0) = \sigma_{\pi N}$ . The standard procedure for its extraction from pion–nucleon ( $\pi N$ ) scattering relies on the low-energy theorem [1, 2]

$$F_\pi^2 \bar{D}^+(s = m^2, t = 2M_\pi^2) = \sigma(2M_\pi^2) + \Delta_R, \quad (1.2)$$

which relates the Born-term-subtracted isoscalar  $\pi N$  scattering amplitude at the Cheng–Dashen point  $\bar{D}^+(s = m^2, t = 2M_\pi^2)$  to the scalar form factor evaluated at  $2M_\pi^2$ . The remainder  $\Delta_R$  is free of chiral logarithms at full one-loop order in chiral perturbation theory (ChPT) [3, 4], and has been estimated as [3]

$$|\Delta_R| \lesssim 2 \text{ MeV}. \quad (1.3)$$

Rewriting (1.2) in terms of

$$\Delta_D = F_\pi^2 \left\{ \bar{D}^+(s = m^2, t = 2M_\pi^2) - d_{00}^+ - 2M_\pi^2 d_{01}^+ \right\}, \quad \Delta_\sigma = \sigma(2M_\pi^2) - \sigma_{\pi N}, \quad (1.4)$$

the extraction of the  $\sigma$  term reduces to the determination of the subthreshold parameters  $d_{00}^+$  and  $d_{01}^+$  as well as the combination  $\Delta_D - \Delta_\sigma - \Delta_R$ . The first two corrections can be calculated using a dispersive approach [5]

$$\Delta_D - \Delta_\sigma = (-3.3 \pm 0.2) \text{ MeV}, \quad (1.5)$$

where the error only covers the uncertainties in the  $\pi\pi$  phase shifts available at that time. Here, we update the determination of  $\Delta_D$  and  $\Delta_\sigma$  using modern  $\pi\pi$  phases, fully including  $\bar{K}K$  intermediate states, and carefully studying the dependence of the results on  $\pi N$  subthreshold parameters as well as the  $\pi N$  coupling constant.

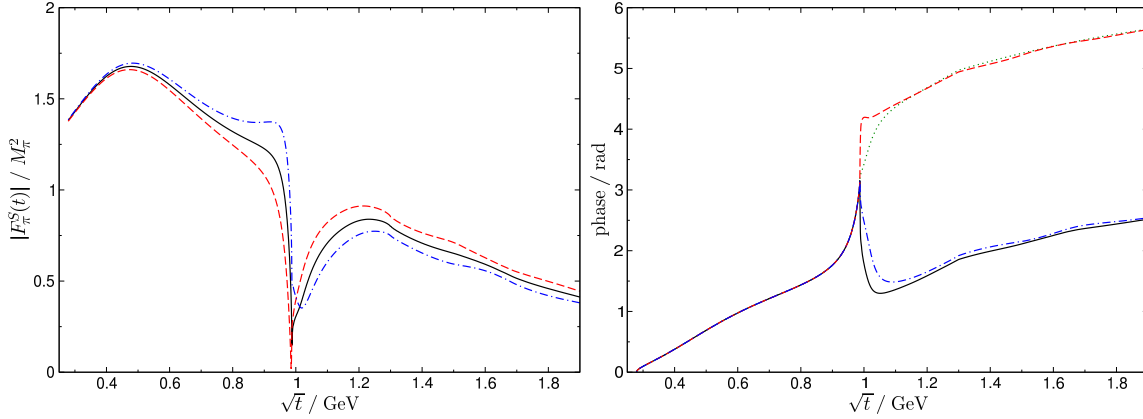
## 2. Scalar pion and kaon form factors

We first consider the case of the scalar pion and kaon form factors  $F_\pi^S(t)$  and  $F_K^S(t)$ , which serve both to illustrate the method and as input for the scalar form factor of the nucleon. Unitarity in the  $\pi\pi/\bar{K}K$  system intertwines both form factors according to [6]

$$\text{Im} F^S(t) = (T(t))^* \Sigma(t) F^S(t), \quad F^S(t) = \begin{pmatrix} F_\pi^S(t) \\ \frac{2}{\sqrt{3}} F_K^S(t) \end{pmatrix}, \quad (2.1)$$

with the phase-space factor

$$\Sigma(t) = \text{diag} \left( \sigma_i^\pi \theta(t - t_\pi), \sigma_i^K \theta(t - t_K) \right), \quad \sigma_i^j = \sqrt{1 - \frac{t_i}{t}}, \quad t_i = 4M_i^2 \quad i \in \{\pi, K\}, \quad (2.2)$$



**Figure 1:** Modulus and phase of the scalar pion form factor. The solid, dashed, and dot-dashed lines refer to  $F_K^S(0) = M_\pi^2/2, 0.4M_\pi^2$ , and  $0.6M_\pi^2$ . The phase of  $F_\pi^S(t)$  is compared to  $\delta_0^0$ , as indicated by the dotted line.

and the  $T$ -matrix

$$T(t) = \begin{pmatrix} \frac{\eta_0^0(t)e^{2i\delta_0^0(t)} - 1}{2i\sigma_t^\pi} & |g(t)|e^{i\psi_0^0(t)} \\ |g(t)|e^{i\psi_0^0(t)} & \frac{\eta_0^0(t)e^{2i(\psi_0^0(t) - \delta_0^0(t))} - 1}{2i\sigma_t^K} \end{pmatrix}, \quad (2.3)$$

expressed in terms of the  $\pi\pi$  and  $\pi\pi \rightarrow \bar{K}K$  phase shifts  $\delta_0^0$  and  $\psi_0^0$  as well as the inelasticity parameter  $\eta_0^0 = \sqrt{1 - 4\sigma_t^\pi \sigma_t^K |g(t)|^2 \theta(t - t_\pi)}$ . The two-channel Muskhelishvili–Omnès (MO) problem [7, 8] defined by the unitarity relation (2.1) permits two linearly independent solutions  $\Omega_1, \Omega_2$  [7], which may be combined in the Omnès matrix  $\Omega(t)$ . In general, there is no analytical solution for  $\Omega(t)$ , we follow here the discretization method of [9] for its numerical calculation.

Since the form factors are devoid of a left-hand cut, they are related directly to the solutions of the MO problem with coefficients determined by  $F_\pi^S(0)$  and  $F_K^S(0)$  [6]. Using ChPT at  $\mathcal{O}(p^4)$  and the low-energy constants from [10] we find

$$F_\pi^S(0) = (0.984 \pm 0.006)M_\pi^2, \quad F_K^S(0) = (0.4 \dots 0.6)M_\pi^2, \quad (2.4)$$

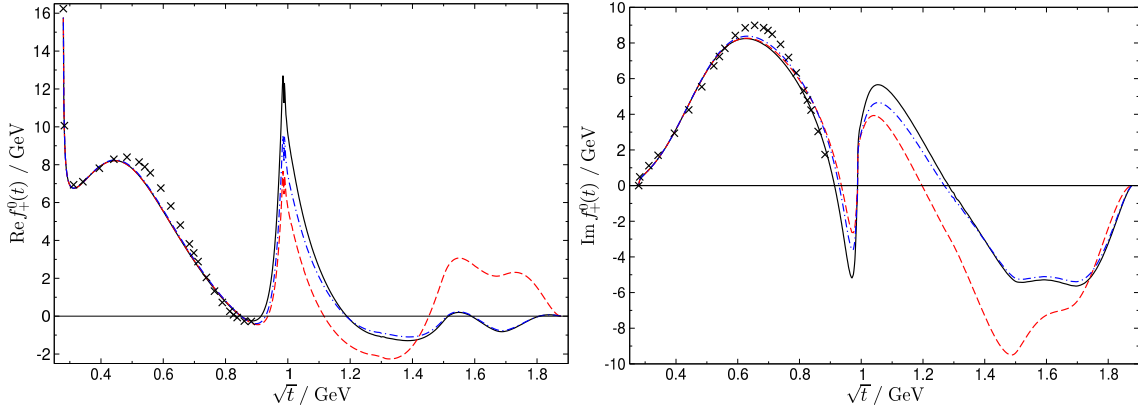
which, together with  $\delta_0^0$  and  $\eta_0^0$  from an extended Roy-equation analysis of  $\pi\pi$  scattering [11],  $\psi_0^0$  from partial-wave analyses [12], and  $|g(t)|$  from a Roy–Steiner (RS) analysis of  $\pi K$  scattering [13], yield the results for  $F_\pi^S(t)$  depicted in Fig. 1. The strong dependence of  $F_\pi^S(t)$  near  $t_K$  on  $F_K^S(0)$  attests to the inherent two-channel nature of the problem and implies that an effective single-channel description in terms of the phase of  $F_\pi^S(t)$  only works for sufficiently large  $F_K^S(0)$ .

### 3. From Roy–Steiner equations to the scalar form factor

Unitarity couples the  $\pi\pi \rightarrow \bar{N}N$  and  $\bar{K}K \rightarrow \bar{N}N$   $S$ -waves  $f_+^0(t)$  and  $h_+^0(t)$  analogously to (2.1)

$$\text{Im} f(t) = (T(t))^* \Sigma(t) f(t), \quad f(t) = \begin{pmatrix} f_+^0(t) \\ \frac{2}{\sqrt{3}} h_+^0(t) \end{pmatrix}, \quad (3.1)$$

but due to the presence of the left-hand cut the solution of the corresponding MO problem involves inhomogeneous contributions, which may be derived from RS equations, cf. [13–15]. Generically,



**Figure 2:** Results for the real and imaginary part of  $f_+^0(t)$ . The solid, dashed, and dot-dashed lines refer to the input RS1, RS2, and RS3 as described in the main text. The black crosses indicate the results of [17].

the integral equation takes the form

$$f(t) = \Delta(t) + (a + bt)(t - 4m^2) + \frac{t^2(t - 4m^2)}{\pi} \int_{t_\pi}^{\infty} dt' \frac{\text{Im} f(t')}{t'^2(t' - 4m^2)(t' - t)}, \quad (3.2)$$

where  $\Delta(t)$  includes Born terms,  $s$ -channel integrals, and higher  $t$ -channel partial waves, while  $a$  and  $b$  subsume subthreshold parameters that emerge as subtraction constants. The main difficulty in the evaluation of the formal solution

$$f(t) = \Delta(t) + (t - 4m^2)\Omega(t)(\mathbb{1} - t\dot{\Omega}(0))a + t(t - 4m^2)\Omega(t)b - \frac{t^2(t - 4m^2)}{\pi} \Omega(t) \int_{t_\pi}^{t_m} dt' \frac{\text{Im} \Omega^{-1}(t') \Delta(t')}{t'^2(t' - 4m^2)(t' - t)} + \frac{t^2(t - 4m^2)}{\pi} \Omega(t) \int_{t_m}^{\infty} dt' \frac{\Omega^{-1}(t') \text{Im} f(t')}{t'^2(t' - 4m^2)(t' - t)}, \quad (3.3)$$

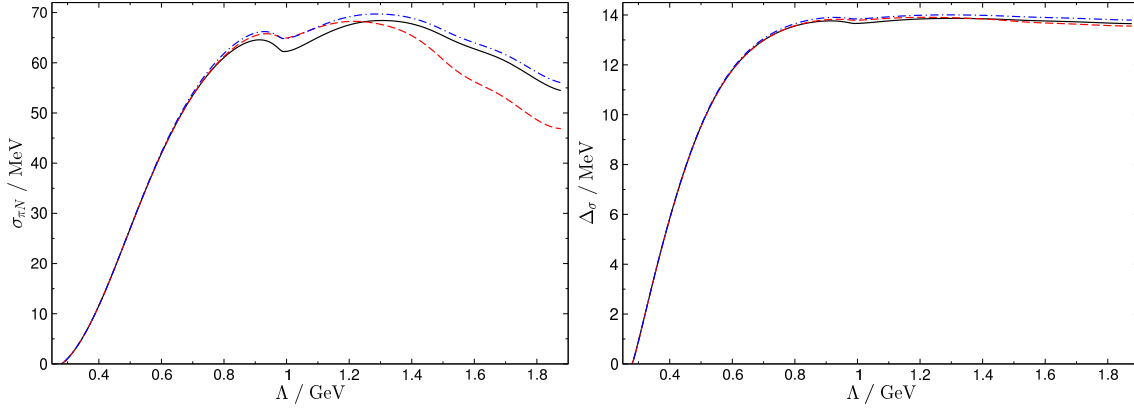
concerns the construction of the Omnès matrix for a finite matching point  $t_m$  [14].

In the numerical analysis we put  $\text{Im} f(t) = 0$  above  $t_m$ , which we choose as  $t_m = 4m^2$  (thus exploiting a kinematical zero of  $f(t)$ ), take the  $\pi N$  and  $KN$   $s$ -channel partial waves from [16], and use the KH80  $\pi N$  coupling constant and subthreshold parameters as reference point [17]. In order to assess the uncertainties for higher energies we consider the following variants of the input: first, we keep the phase shifts  $\delta_0^0$  and  $\psi_0^0$  constant above  $\sqrt{t_0} = 1.3 \text{ GeV}$  (“RS1”), where  $4\pi$  intermediate states become important and thus the two-channel approximation will break down, and second, guide both phase shifts smoothly to their asymptotic value of  $2\pi$  as for the meson form factors (“RS2”). Finally, we amend RS1 in such a way that  $\Delta_2(t)$ , the  $KN$  component of the inhomogeneity, is put to zero in order to assess the uncertainty in the  $KN$  input (“RS3”). The corresponding results for  $f_+^0(t)$  depicted in Fig. 2 show that the largest uncertainty is induced by the high-energy phase shifts.

## 4. Results

The scalar form factor of the nucleon fulfills the unitarity relation

$$\text{Im} \sigma(t) = \frac{2}{4m^2 - t} \left\{ \frac{3}{4} \sigma_t^\pi (F_\pi^S(t))^* f_+^0(t) \theta(t - t_\pi) + \sigma_t^K (F_K^S(t))^* h_+^0(t) \theta(t - t_K) \right\}, \quad (4.1)$$



**Figure 3:**  $\sigma_{\pi N}$  and  $\Delta_\sigma$  as a function of the integral cutoff  $\Lambda$ .

so that, based on the results of the previous sections, the un- and once-subtracted dispersion relations

$$\sigma(t) = \frac{1}{\pi} \int_{t_\pi}^{\infty} dt' \frac{\text{Im} \sigma(t')}{t' - t} = \sigma_{\pi N} + \frac{t}{\pi} \int_{t_\pi}^{\infty} dt' \frac{\text{Im} \sigma(t')}{t'(t' - t)} \quad (4.2)$$

evaluated at  $t = 0$  and  $t = 2M_\pi^2$  in principle determine  $\sigma_{\pi N}$  and  $\Delta_\sigma$  provided the two-channel approximation for the spectral function is sufficiently accurate in the energy range dominating the dispersive integral. Fig. 3 shows that, while the dispersion relation converges too slowly for the  $\sigma$  term itself, the result for  $\Delta_\sigma$  becomes stable for  $\Lambda \gtrsim 1$  GeV. Adding the uncertainties from the spectral function and the variation of the integral cutoff between  $\Lambda = 1.3$  GeV and  $\Lambda = 2m$ , we find

$$\begin{aligned} \Delta_\sigma &= (13.9 \pm 0.3) \text{ MeV} \\ &+ Z_1 \left( \frac{g^2}{4\pi} - 14.28 \right) + Z_2 \left( d_{00}^+ M_\pi + 1.46 \right) + Z_3 \left( d_{01}^+ M_\pi^3 - 1.14 \right) + Z_4 \left( b_{00}^+ M_\pi^3 + 3.54 \right), \\ Z_1 &= 0.36 \text{ MeV}, \quad Z_2 = 0.57 \text{ MeV}, \quad Z_3 = 12.0 \text{ MeV}, \quad Z_4 = -0.81 \text{ MeV}, \end{aligned} \quad (4.3)$$

where we have made the dependence on the  $\pi N$  parameters explicit (note that more modern determinations point to lower values of the  $\pi N$  coupling constant around  $g^2/4\pi \sim 13.7$  [18–20]).

Similarly, the correction  $\Delta_D$  follows from the  $t$ -channel expansion

$$\bar{D}^+(s = m^2, t) = d_{00}^+ + d_{01}^+ t - 16t^2 \int_{t_\pi}^{\infty} dt' \frac{\text{Im} f_+^0(t')}{t'^2(t' - 4m^2)(t' - t)} + \{J \geq 2\} + \{s\text{-channel integrals}\} \quad (4.4)$$

evaluated at  $t = 2M_\pi^2$ , which gives

$$\begin{aligned} \Delta_D &= (12.1 \pm 0.3) \text{ MeV} \\ &+ \tilde{Z}_1 \left( \frac{g^2}{4\pi} - 14.28 \right) + \tilde{Z}_2 \left( d_{00}^+ M_\pi + 1.46 \right) + \tilde{Z}_3 \left( d_{01}^+ M_\pi^3 - 1.14 \right) + \tilde{Z}_4 \left( b_{00}^+ M_\pi^3 + 3.54 \right), \\ \tilde{Z}_1 &= 0.42 \text{ MeV}, \quad \tilde{Z}_2 = 0.67 \text{ MeV}, \quad \tilde{Z}_3 = 12.0 \text{ MeV}, \quad \tilde{Z}_4 = -0.77 \text{ MeV}. \end{aligned} \quad (4.5)$$

Comparison with (4.3) shows that the dependence on the  $\pi N$  parameters cancels nearly completely in the difference

$$\Delta_D - \Delta_\sigma = (-1.8 \pm 0.2) \text{ MeV}. \quad (4.6)$$

This cancellation can be explained by the observation that the spectral function in both dispersion relations involves  $f_+^0$  in a very similar manner, so that both integrals are equally affected by the dependence on the  $\pi N$  parameters inherited from  $f_+^0$ . In the same way, part of the uncertainties discussed in Sect. 3 drop out, so that the final error estimate for  $\Delta_D - \Delta_\sigma$  even decreases compared to the uncertainty in both corrections individually.

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